Ex 1 Throughout this problem, we'll use techniques that we already know to create successively better approximations for the function $f(x) = e^x$.

(a) Come up with the constant function (also known as a degree 0 polynomial), $g_0(x)$, that passes through the same point as $f(x)$ at $x = 0$. Graph $g_0(x)$ on the following axes.

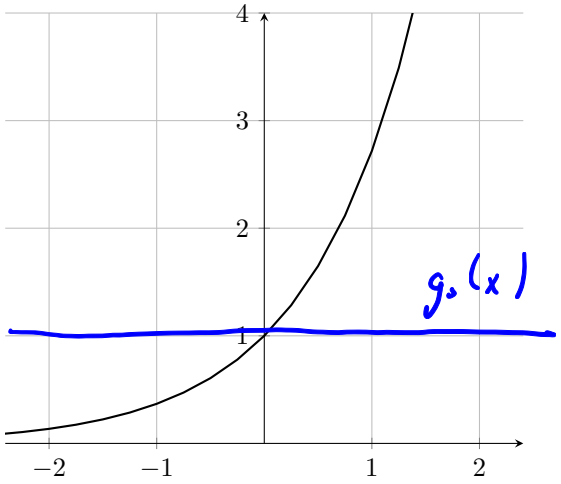
① what pt. does $f(x)$ pass through at $x = 0$?

↳ $f(0) = e^0 = 1$

↳ $(0, 1)$

$g_0(x) = 1$

i. Graph $g_0(x)$ on the following axes



ii. Is it possible for $g_0(x)$ to have the same slope as $f(x)$ at $x = 0$? Why or why not?

No - $g_0(x)$ has slope 0

$f(x)$ has nonzero slope

(b) Come up with the linear function (also known as a degree 1 polynomial), $g_1(x)$, that passes through the same point *and* has the same slope as $f(x)$ at $x = 0$ (this is the tangent line!).

$f(x) = e^x$

$f'(x) = e^x$

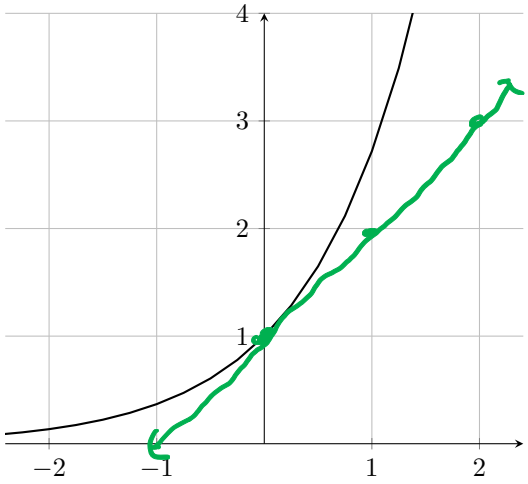
$f'(0) = e^0 = 1$

$g_1(x) = ax + b$

$g_1'(x) = a$

$g_1(x) = x + 1$

i. Graph $g_1(x)$ on the following axes.



ii. Is it possible for $g_1(x)$ to have the same second derivative as $f(x)$ at $x = 0$? Why or why not?

$f''(x) = e^x$

$f''(0) = 1$

$g_1''(x) = 0$

No

iii. How does the constant term of $g_1(x)$ compare to the constant term of $g_0(x)$?

↳ the same

(c) Come up with the quadratic function (a degree 2 polynomial), $g_2(x)$, that passes through the same point, has the same slope (first derivative), and the same concavity (second derivative) as $f(x)$ at $x = 0$.

$f(0) = 1$ → $c = 1$

$f'(0) = 1$ → $b = 1$

$f''(0) = 1$ → $a = \frac{1}{2}$

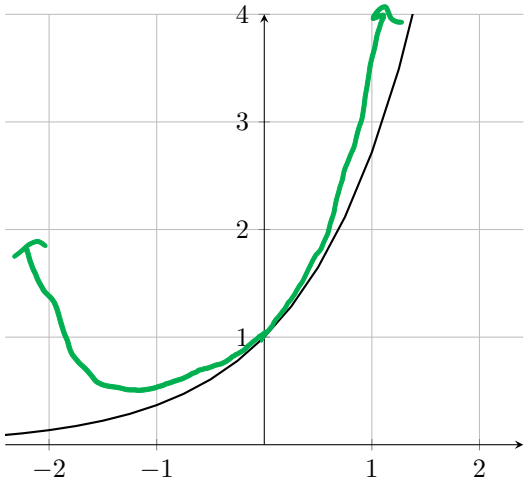
$g_2(x) = ax^2 + bx + c$

$g_2'(x) = 2ax + b$

$g_2''(x) = 2a$

$g_2(x) = \frac{1}{2}x^2 + x + 1$

i. Graph $g_2(x)$ on the following axes.



ii. Is it possible for $g_2(x)$ to have the same third derivative as $f(x)$ at $x = 0$?

$g_2'''(x) = 0$

$f'''(x) = e^x$

→ No

iii. How do the constant and degree 1 terms of $g_2(x)$ compare to the constant and degree 1 terms of $g_1(x)$?

The same

(d) Come up with the cubic (degree 3) polynomial, $g_3(x)$, that passes through the same point, has the same first derivative, second derivative, and third derivative as $f(x)$ at $x = 0$.

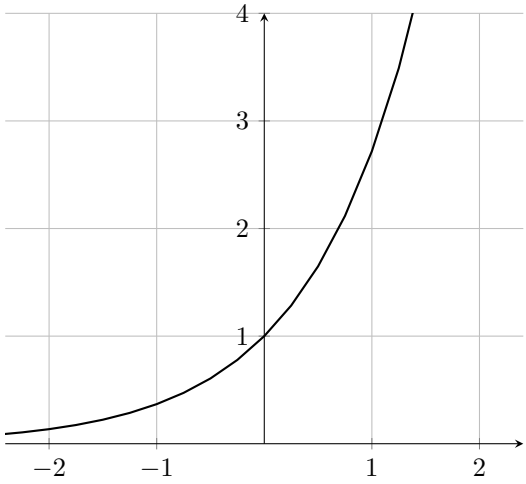
$$g_3(x) = \frac{x^3}{3\cdot 2\cdot 1} + \frac{x^2}{2\cdot 1} + x + 1$$

$$g_3'(x) = \frac{3x^2}{3\cdot 2\cdot 1} + \frac{2x^1}{2\cdot 1} + 1$$

$$g_3''(x) = \frac{2x^1}{2\cdot 1} + 1$$

$$g_3'''(x) = 1$$

i. Graph $g_3(x)$ on the following axes.



ii. Is it possible for $g_3(x)$ to have the same fourth derivative as $f(x)$ at $x = 0$?

No

iii. How do the constant, degree 1, and degree 2 terms of $g_3(x)$ compare to the corresponding terms of $g_2(x)$?

The same

(e) Come up with the degree n polynomial, $g_n(x)$, that passes through the same point, has the same first derivative, second derivative,...,nth derivative as $f(x)$ at $x = 0$. The following notation might be helpful: $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.

$$g_0(x) = 1$$

$$g_1(x) = 1 + x$$

$$g_2(x) = 1 + x + \frac{x^2}{2}$$

$$g_3(x) = 1 + \frac{x}{1} + \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2 \cdot 1}$$

$$\vdots$$

$$g_n(x) = 1 + \frac{x}{1} + \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2 \cdot 1} + \cdots + \frac{x^n}{n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1}$$

$$= x^0 + \frac{x}{1} + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

n factorial

$$\downarrow$$

$$g_\infty(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\lim_{t \to \infty} \arctan(t)$$

Factorials

$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$1! = 1$

$0! = 1$

↑

Why should this be true?

(A1) $n! = n \cdot (n-1)(n-2) \cdots 2 \cdot 1$

$= n \cdot (n-1)!$

$n=1 \rightarrow 1! = 1 \cdot (0!)$

$1 = 0!$

(A2) $n! = \underbrace{n(n-1) \cdots 2 \cdot 1}$

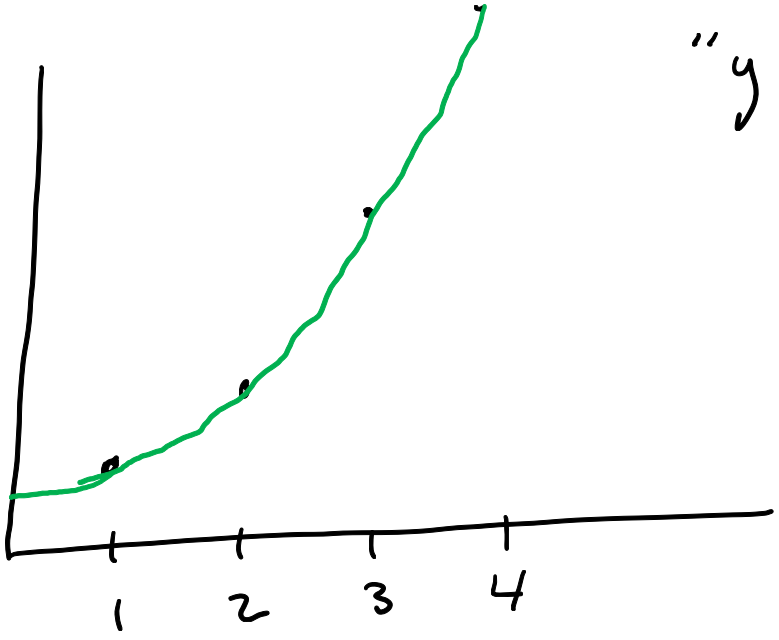
multiply n numbers together

$1 = 0!$ = multiply 0 numbers together

2^n = multiply n numbers together

$1 = 2^0$ = multiply 0 numbers together

A3



"y = x!"

Gamma → $\Gamma(x) = \int_{0,}^{x}$ ~~

→ $\Gamma(n) \approx n!$

$\Gamma(0) = 1$